MATH 3060 : HW6 Solution

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(1) Let $f : \mathbb{R} \to \mathbb{R}$ be C^2 and $f(x_0) = 0$, $f'(x_0) \neq 0$. Show that there exists $\rho > 0$ such that

$$Tx = x - \frac{f(x)}{f'(x)}, \quad x \in (x_0 - \rho, x_0 + \rho)$$

is a contraction. (This is the Newton's method.)

Sol) As $f'(x_0) \neq 0$, there exists $\delta > 0$ such that for any $x \in B_\delta(x_0)$, $f'(x) \neq 0$.

Then $\tilde{T} : B_\delta(x_0) \to \mathbb{R}$ defined as $\tilde{T}x := x - \frac{f(x)}{f'(x)}$ is well-defined.

Since f is C^2, T is differentiable with

$$\tilde{T}'(x) = 1 - \frac{(f'(x))^2 - f(x) f''(x)}{(f'(x))^2} = \frac{f(x) f''(x)}{(f'(x))^2},$$

which is continuous.

Hence, $\tilde{T}$ is C^1 with $\tilde{T}x_0 = x_0 + 0 = x_0$; Also, $\tilde{T}'(x_0) = 0$.

By the continuity of $\tilde{T}'$ at x_0, there exists $\frac{\delta}{2} > \rho > 0$ such that for any $x \in B_{2\rho}(x_0)$, $|\tilde{T}'(x)| < 1$.

Define $T := \tilde{T}|_{B_\rho(x_0)} : B_\rho(x_0) \to \mathbb{R}$. Then $Tx_0 = x_0$ and $T'(x_0) = 0$.

Showing $T: B_\rho(x_0) \longrightarrow \mathbb{R}$ is a contraction:

Choose $\gamma := \max_{x \in \overline{B_\rho(x_0)}} |T'(x)| < 1$, then for any $x, x' \in B_\rho(x_0)$,

$$|Tx - Tx'| = |T'(\xi)|\,|x - x'|, \quad \left(\begin{array}{l}\text{where } \xi \text{ is between } x \text{ and } x' \\ \text{by Mean Value Theorem of } T \text{ on } \overline{B_\rho}(x_0)\end{array}\right)$$

$$\leq \gamma \cdot |x - x'|$$

In particular, choose $x' = x_0$, then for any $x \in B_\rho(x_0)$,

$$|Tx - x_0| = |Tx - Tx_0| \leq \gamma\,|x - x_0| < 1 \cdot \rho = \rho$$

$\therefore$ $T(B_\rho(x_0)) \subseteq B_\rho(x_0)$, and $T: B_\rho(x_0) \longrightarrow B_\rho(x_0)$ is a contraction.

(2) Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{2}x + x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Show that f is differentiable at $x=0$ with $f'(0) = \frac{1}{2}$, but it has no local inverse at $x=0$. Does it contradict the inverse function theorem?

Sol) Computing the derivative of f:

For $x \neq 0$, $f(x)$ is clearly differentiable with

$$f'(x) = \frac{1}{2} + 2x \sin\frac{1}{x} + x^2 \cdot \cos\frac{1}{x} \cdot \left(-\frac{1}{x^2}\right) = \frac{1}{2} + 2x\sin\frac{1}{x} - \cos\frac{1}{x}.$$

For $x=0$, note that for any $y \neq 0$, $\dfrac{f(y)-f(0)}{y-0} = \dfrac{1}{2} + y \sin \dfrac{1}{y}$.

$\therefore \displaystyle\lim_{y \to 0} \frac{f(y)-f(0)}{y-0} = \frac{1}{2}$ exists. Hence, f is differentiable with $f'(0) = \frac{1}{2}$.

Showing f has no local inverse at $x=0$: Suppose on the contrary, there exists an open interval I containing 0 such that $f|_I : I \to f(I)$ has inverse. In particular, f is injective.

Recall that any injective continuous function on an interval is monotonic.

(Pf: Exercise. Hint: Proof by contradiction using Intermediate Value Theorem)

In particular, $f|_I$ is monotonic. Since $f|_I$ is also differentiable,

either for any $x \in I$, $f'(x) \geq 0$ or for any $x \in I$, $f'(x) \leq 0$.

Meanwhile, choose $k \in \mathbb{N}$ large enough such that $\frac{1}{2k\pi} \in I$.

Then $f'\left(\frac{1}{2k\pi}\right) = \frac{1}{2} + 0 - 1 = -\frac{1}{2}$; $f'\left(\frac{1}{2k\pi + \pi}\right) = \frac{1}{2} + 0 + 1 = \frac{3}{2}$.

This leads to contradiction. Therefore, f has no local inverse at $x = 0$.

This question does not contradict with the Inverse Function Theorem

because f is not C^1 at $x = 0$: $\lim_{x \to 0} f'(x)$ does not exist, as $\lim_{x \to 0} \cos \frac{1}{x}$ does not exist.

(3) Let $a > 0$, define a mapping $T: C[-a,a] \to C[-a,a]$ by

$$Tx(t) = 1 + \int_0^t x(s)\,ds.$$

Let $x(t) \equiv 1$ on $[-a, a]$

Find $T^n x$, $\forall n \geq 0$. Does $\{T^n x\}$ converge in $(C[-a,a], d_\infty)$? If so, what is the limit?

Sol) Computing $T^n x$ for small $n \geq 0$: $x(t) \equiv 1$; $Tx(t) = 1 + \int_0^t 1\,ds = 1 + t$;

$(T^2 x)(t) = T(Tx)(t) = 1 + \int_0^t (1+s)\,ds = 1 + [s + \frac{s^2}{2}]_0^t = 1 + t + \frac{t^2}{2}$.

By tracing pattern, we have $(T^n x)(t) = \sum_{k=0}^{n} \frac{t^k}{k!}$.

Showing $(T^n x)(t) = \sum_{k=0}^{n} \frac{t^k}{k!}$ for any $n \geq 0$ by induction:

Base step: $[n=0]$ case holds by definition.

Inductive step: Assume $(T^N x)(t) = \sum_{k=0}^{N} \frac{t^k}{k!}$ for some $N \in \mathbb{N} \cup \{0\}$.

Showing $(T^{N+1} x)(t) = \sum_{k=0}^{N+1} \frac{t^k}{k!}$:

$(T^{N+1} x)(t) = T(T^N x)(t) = 1 + \int_0^t \sum_{k=0}^{N} \frac{s^k}{k!}\,ds = 1 + \sum_{k=0}^{N} \left[\frac{s^{k+1}}{(k+1)!}\right]_0^t = \sum_{k=0}^{N+1} \frac{t^k}{k!}$.

$\therefore$ By Induction, $(T^n x)(t) = \sum_{k=0}^{n} \frac{t^k}{k!}$ for any $n \geq 0$.

In this case, $\left\{ \sum_{k=0}^{n} \frac{t^k}{k!} \right\}_{n=0}^{\infty}$ converges uniformly to $\sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t$ on $[-a,a]$.

Therefore, $\{T^n x\}_{n=0}^{\infty}$ converges to e^t in $(C[-a,a], d_\infty)$.

Rmk For $0 < a < 1$, there is an alternative proof of $\lim_{n\to\infty} (T^n x)(t) = e^t$ via the Contraction Mapping Principle as follows:

To show $\{T^n x\}_{n=1}^{\infty}$ converges, by the proof of Contraction Mapping Principle, it suffices to show that $T: (C[-a,a], d_\infty) \to (C[-a,a], d_\infty)$ is a contraction.

For any $x_1, x_2 \in (C[-a,a], \|\cdot\|_\infty)$, for any $t \in [-a,a]$,

$$|(Tx_2)(t) - (Tx_1)(t)| = \left| \left(1 + \int_0^t x_2(s)\,ds\right) - \left(1 + \int_0^t x_1(s)\,ds\right) \right|$$

$$= \left| \int_0^t (x_2(s) - x_1(s))\,ds \right| \leq \|x_2 - x_1\|_\infty |t| \leq a \|x_2 - x_1\|_\infty$$

$\therefore$ $\|Tx_2 - Tx_1\|_\infty \leq a \|x_2 - x_1\|_\infty$, where $a < 1$ by assumption, hence

$T: (C[-a,a], d_\infty) \to (C[-a,a], d_\infty)$ is a contraction on a complete metric space $(C[-a,a], d_\infty)$.

By the proof of Contraction Mapping Principle, $\{T^n x\}_{n=1}^{\infty}$ converges to the unique fixed point $y(t)$ of T, i.e. $y(t) = 1 + \int_0^t y(s)\,ds$.

$\therefore$ It suffices to show that $y(t) = e^t$.

Solving the integral equation $y(t) = 1 + \int_0^t y(s)\,ds$:

Since $y(t)$ is continuous, by Fundamental Theorem of Calculus,

$1 + \int_0^t y(s)\,ds$ is C^1 with $\left(1 + \int_0^t y(s)\,ds\right)'(t) = y(t)$

$\therefore\ y'(t) = \left(1 + \int_0^t y(s)\,ds\right)'(t) = y(t)$.

$\therefore\ \frac{d}{dt}\left(\log(y(t))\right) = 1$, hence $y(t) = e^{t+c}$, for some $c \in \mathbb{R}$.

Substituting $t=0$ into $y(t) = 1 + \int_0^t y(s)\,ds$ yields $y(0) = 1$.

$\therefore\ c=0$, hence $y(t) = e^t$.

$\therefore\ \lim_{n\to\infty} (T^n x)(t) = e^t$.

(4) Let $a>0$, define a mapping $T: C[-a,a] \to C[-a,a]$ by

$$Tx(t) = 1 + \int_0^t s\, x(s)\, ds.$$

Let $x(t) \equiv 1$ on $[-a, a]$

Find $T^n x$, $\forall n \geq 0$. Does $\{T^n x\}$ converge in $(C[-a,a], d_\infty)$? If so, what is the limit?

Sol) Computing $T^n x$ for small $n \geq 0$: $x(t) \equiv 1$; $Tx(t) = 1 + \int_0^t s\,ds = 1 + \frac{t^2}{2}$;

$(T^2 x)(t) = T(Tx)(t) = 1 + \int_0^t s(1 + \frac{s^2}{2})\,ds = 1 + [\frac{s^2}{2} + \frac{s^4}{2\cdot 4}]_0^t = 1 + \frac{t^2}{2} + \frac{t^4}{2\cdot 4}$

By tracing pattern, we have $(T^n x)(t) = \sum_{k=0}^{n} \frac{t^{2k}}{2^k \cdot k!}$.

showing $(T^n x)(t) = \sum_{k=0}^{n} \frac{t^{2k}}{2^k \cdot k!}$ for any $n \geq 0$ by induction:

Base step: $[n=0]$ case holds by definition.

Inductive step: Assume $(T^N x)(t) = \sum_{k=0}^{N} \frac{t^{2k}}{2^k \cdot k!}$ for some $N \in \mathbb{N} \cup \{0\}$.

Showing $(T^{N+1} x)(t) = \sum_{k=0}^{N+1} \frac{t^{2k}}{2^k \cdot k!}$: $(T^{N+1} x)(t) = T(T^N x)(t) = 1 + \int_0^t s \sum_{k=0}^{N} \frac{s^{2k}}{2^k \cdot k!}\,ds$

$= 1 + \sum_{k=0}^{N} \left[\frac{s^{2k+2}}{2^{k+1}\cdot(k+1)!}\right]_0^t = \sum_{k=0}^{N+1} \frac{t^{2k}}{2^k \cdot k!}$.

$\therefore$ By Induction, $(T^n x)(t) = \sum_{k=0}^{n} \frac{t^{2k}}{2^k \cdot k!}$ for any $n \geq 0$.

In this case, $\left\{\sum_{k=0}^{n} \frac{t^{2k}}{2^k \cdot k!}\right\}_{n=0}^{\infty}$ converges uniformly to $\sum_{k=0}^{\infty} \frac{t^{2k}}{2^k \cdot k!} = \sum_{k=0}^{\infty} \frac{\left(\frac{t^2}{2}\right)^k}{k!} = e^{\frac{t^2}{2}}$ on $[-a,a]$.

Therefore, $\{T^n x\}_{n=0}^{\infty}$ converges to $e^{\frac{t^2}{2}}$ in $(C[-a,a], d_\infty)$.

Rmk For $0 < a < \sqrt{2}$, there is an alternative proof of $\lim_{n\to\infty} (T^n x)(t) = e^{\frac{t^2}{2}}$

via the Contraction Mapping Principle as follows:

To show $\{T^n x\}_{n=1}^{\infty}$ converges, by the proof of Contraction Mapping Principle,

it suffices to show that $T : (C[-a,a], d_\infty) \longrightarrow (C[-a,a], d_\infty)$ is a contraction.

For any $x_1, x_2 \in (C[-a,a], \|\cdot\|_\infty)$, for any $t \in [-a,a]$,

$$|(Tx_2)(t) - (Tx_1)(t)| = \left|\left(1 + \int_0^t s x_2(s)\,ds\right) - \left(1 + \int_0^t s x_1(s)\,ds\right)\right|$$

$$= \left|\int_0^t s(x_2(s) - x_1(s))\,ds\right| \leqslant \|x_2 - x_1\|_\infty \cdot \left(\frac{t^2}{2}\right) \leqslant \frac{a^2}{2} \cdot \|x_2 - x_1\|_\infty$$

$\therefore$ $\|Tx_2 - Tx_1\|_\infty \leqslant \frac{a^2}{2} \|x_2 - x_1\|_\infty$, where $\frac{a^2}{2} < 1$ by assumption, hence

$T : (C[-a,a], d_\infty) \longrightarrow (C[-a,a], d_\infty)$ is a contraction on a complete metric space $(C[-a,a], d_\infty)$.

By the proof of Contraction Mapping Principle, $\{T^n x\}_{n=1}^{\infty}$ converges to the unique

fixed point $y(t)$ of T, i.e. $y(t) = 1 + \int_0^t s\,y(s)\,ds$.

$\therefore$ It suffices to show that $y(t) = e^{\frac{t^2}{2}}$.

Solving the integral equation $y(t) = 1 + \int_0^t s y(s)\, ds$:

Since $t y(t)$ is continuous, by Fundamental Theorem of Calculus,

$1 + \int_0^t s y(s)\, ds$ is C^1 with $\left(1 + \int_0^t s y(s)\, ds\right)'(t) = t y(t)$.

$\therefore\ y'(t) = \left(1 + \int_0^t s y(s)\, ds\right)'(t) = t y(t)$.

$\therefore\ \frac{d}{dt}\left(\log(y(t))\right) = t$, hence $y(t) = e^{\frac{t^2}{2} + C}$, for some $C \in \mathbb{R}$.

Substituting $t = 0$ into $y(t) = 1 + \int_0^t s y(s)\, ds$ yields $y(0) = 1$.

$\therefore\ C = 0$, hence $y(t) = e^{\frac{t^2}{2}}$.

$\therefore\ \lim_{n \to \infty} (T^n x)(t) = e^{\frac{t^2}{2}}$.